Modeling 1

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1 Introduction

The two-body problem describes the solution to the equations of motion of one body orbiting another in a gravitational field. The motion can be described with the equation:

$$\ddot{\mathbf{r}} = -\frac{GM}{r^2}\frac{\vec{r}}{r} \tag{1}$$

Attempting to integrate this twice to solve for $\vec{r}$ results in an elliptical integral. Thus, we must instead solve for the position using the true anomaly:

$$r(\theta) = \frac{a(1-e^2}{1+e\cos\theta} \tag{2}$$

Where θ is the angle swept out by the body in orbit, a is the semi-major axis and e is the eccentricity of the orbit. θ is not implicitly a function of time. However, it is related to time through the mean anomaly and eccentric anomaly. The mean anomaly is given by:

$$M = 2\pi\frac{(t-t_p)\%T}{T} \tag{3}$$

Where t_p is the initial time, in our case 0, and T is the period. The eccentric anomaly is given by:

$$E = M + e\sin E \tag{4}$$

Clearly, it is impossible to explicitly solve for E. To reconcile this we'll solve for it iterative using the Newton-Raphson method. Let f be defined as:

$$f(E) = E - e\sin E - M \tag{5}$$

So that

$$f'(E) = 1 - e\cos E \tag{6}$$

Which gives us the iterative function:

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E} \tag{7}$$

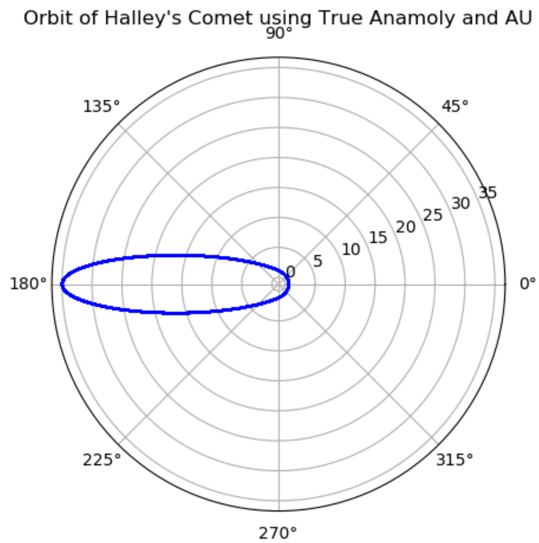

Figure 1: A plot of the orbit of Halley's Comet in polar coordinates, where r is in AU and theta is in degrees.

Thus, using the Newton-Raphson and 7 we are able to calculate E for a given t. Finally, we can solve for θ using:

$$\tan\frac{\theta}{2} = (\frac{1+e}{1-e})^{\frac{1}{2}}\tan\frac{E}{2} \tag{8}$$

Applying this method to solve for the orbit of Halley's Comet yields the orbit radius $\vec{r}$ as a function of θ.

2 Runge-Kutta Approximation

The Runge-Kutta methods are an expansion which mirrors the expansion of the Taylor Series, with the main difference being that the Runge-Kutta methods require one less order than the Taylor Series. They are used to to iteratively solve differential equations. The derivation of the third-order Runge-Kutta methods is:

$$y(x+h) = y(x) + \sum_{i=1}^{n} \gamma_i k_i \tag{9}$$

$$k_i = hf(x + \alpha_i h, y + \sum_{j=1}^{i-1} \beta_{ij} k_j \tag{10}$$

Expanding k_1 and k_2 as we did in class, we obtain:

$$y(x+h) = y(x) + \gamma_1 hf + \gamma_2(hf + \alpha h^2 f_x + \beta h^2 f_y f) \tag{11}$$

Now, we need to expand k_3:

$$k_3 = hf(x + \alpha_2 h, y + \beta_1 hf + \beta_2 k_2) \tag{12}$$

$$\gamma_3 k_3 = \gamma_3 hf(x + \alpha_2 h, y + \beta_1 hf + \beta_2 hf + \beta_2 \alpha_1 h^2 f_x + \beta_2 \beta_1 h^2 f_y f) \tag{13}$$

$$\gamma_3 k_3 = \gamma_3 hf(x + \alpha_2 h, y + h(\beta_1 + \beta_2) + \beta_2 \alpha_1 h^2 f_x + \beta_2 \beta_1 h^2 f_y f) \tag{14}$$

$$\begin{aligned}\gamma_3 k_3 = \gamma_3 h[f + \alpha_2 hf_x + h(\beta_1 + \beta_2) f_x + h(\beta_1 + \beta_2) f_y f + \beta_2 + \alpha_1 h^2 f_{xx} + \\ + \beta_2 \beta_1 h^2 f_{yx} f + \beta_2 \beta_1 h^2 f_y f_x + \beta_2 \alpha_1 h^2 f_{xy} f + \beta_2 \beta_1 f_{yy} f^2 + \beta_2 \beta_1 f_y f_y f]\end{aligned} \tag{15}$$

We then compare the coefficients in $\gamma_1 k_1, \gamma_2 k_2, \gamma_2 k_2$ to the coefficients in the third order Taylor Series:

$$y(x+h) = y(x) + hf(x,y) + \frac{1}{2} h^2 f' + \frac{1}{6} h^3 f'' \tag{16}$$

$$y(x) + hf + \frac{1}{2} h^2 (f_x + f_y f) + \frac{1}{6} h^3 (f_{xx} + f_{xy} f + 2(f_y f_x) + fy^2 f + f_{fx} f + f_{yy} f^2) \tag{17}$$

This yields the following system of equations for the coefficients of the Runge-Kutta method:

$$\gamma_1 + \gamma_2 + \gamma_3 = 1$$
$$\gamma_2 + \alpha_1 + \gamma_3 \alpha_2 + \gamma_3(\beta_1 + \beta_2) = \frac{1}{2}$$
$$\gamma_2 \beta_1 + \gamma_3(\beta_1 + \beta_2) = \frac{1}{2}$$
$$\gamma_3 \beta_2 \alpha_1 = \frac{1}{6}$$
$$\gamma_3 \beta_2 \beta_1 = \frac{1}{6}$$

This system cannot be solved explicitly because there are 7 variables and only 5 equations; thus, 2 must be chosen. In this case, I chose:

$$\gamma_3 = \frac{1}{3}, \beta_2 = \frac{1}{2}$$

And the other coefficients are solved, yielding:

$$\alpha_1 = 1, \beta_1 = 1, \gamma_2 = 0, \gamma_1 = \frac{2}{3}, \alpha_2 = 0$$

3 Differential Solution

A 4^{th} order Runge-Kutta method was used to compute the orbit of Halley's Comet. Because the Newton-Raphson method can be computed to an arbitrarily small tolerance, it was taken to be the "true" orbit of the comet. Interestingly, the orbit calculated by the Runge-Kutta method lagged behind that calculated by the Newton-Raphson method.

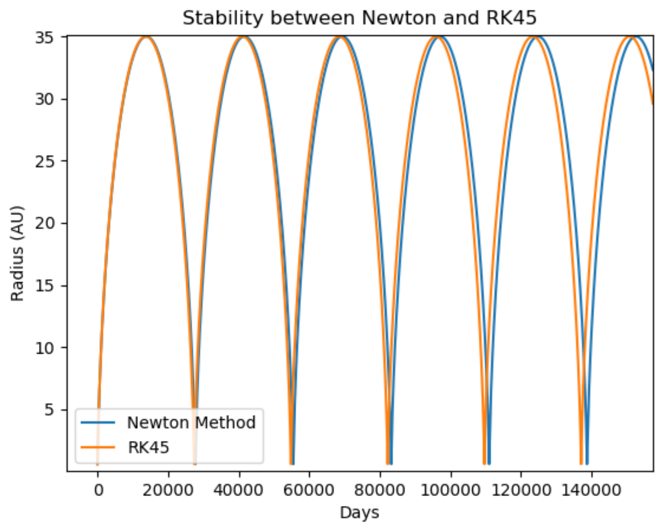

Figure 2: The motion calculated by the Runge-Kutta method lags behind the one calculated by Newton-Raphson

Due to this lag, the error between the RK4 and Newton-Raphson radii is periodic in two ways: during one period of rotation, there is one radius where they are equal, and one where there is a maximum error. Thus, there the time between two zeros and the time between two local maximum values is one orbital period. Second, the error between the two orbits starts at 0 and reaches an absolute maximum of 35.587, the distance between perihelion and aphelion (when the Runge-Kutta method is off by half an orbital period) before returning back to 0.

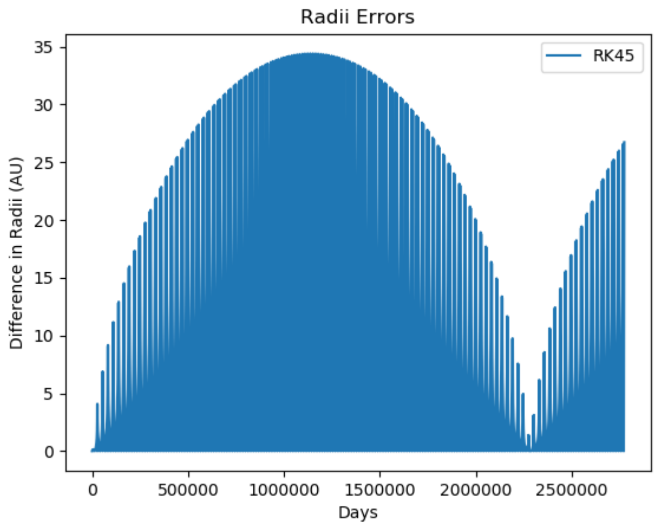

Figure 3: Periodic error created by lag in the RK4 method

The periodicity of the error is also clear in the cumulative sum of the error; on a smaller scale, the error increases more rapidly when the orbits are offset by the largest value in a single orbit, but on a larger scale it increases fastest when the Runge-Kutta Method is around a half-orbit off from the Newton-Raphson method.

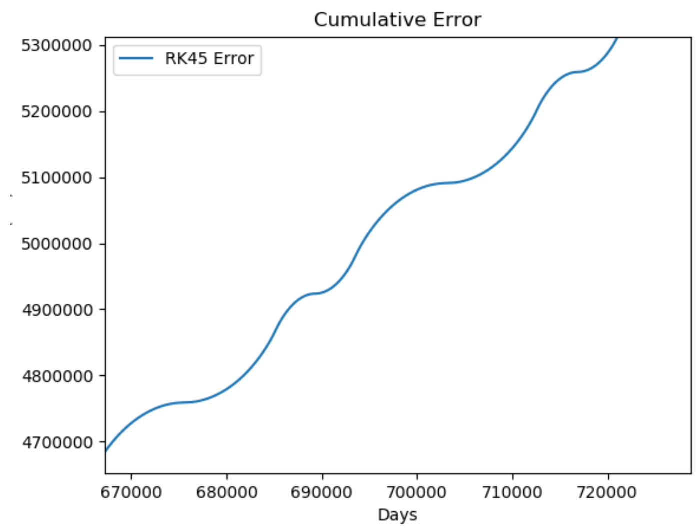

Figure 4: Smaller scale cumulative sum.

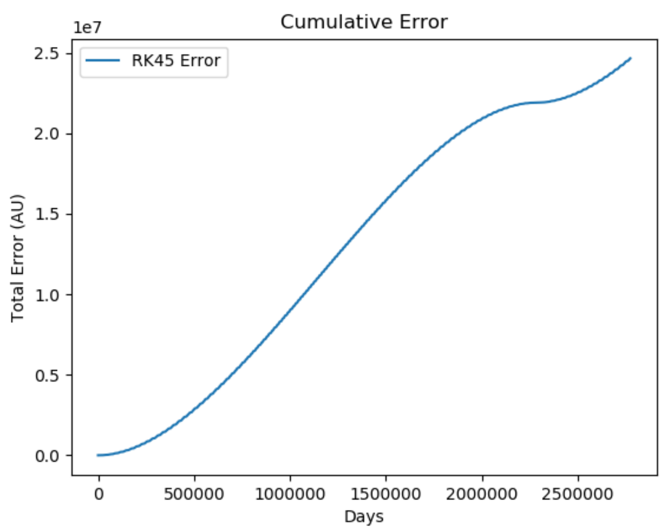

Figure 5: Larger scale cumulative sum.

4 Orbit Stability

The orbit of Halley's Comet was then calculated using the RK4, RK2, and LSODA methods of `scipy.integrate`. On the timescale used, the RK4, RK2, and LSODA orbits of Halley's Comet appear to be stable, outside of the lag they experience relative to the Newton-Raphson solution. They also trace out the same orbit. Due to the absence of complicating factors such as mass loss, rotating coordinate systems, and additional bodies, a stable orbit is not unexpected.

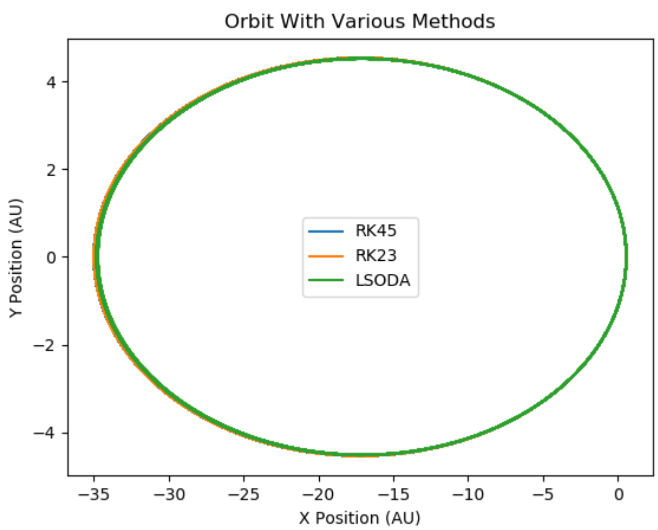

Figure 6: Orbits traced by RK4, RK2 and LSODA methods.

5 Acknowledgements

Rachel Price and I collaborated on the code in an effort to help each other find bugs, resolve errors, and understand and implement off-the-shelf methods.