Modeling 2

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1 Non-inertial Forces and Potentials

Solving for the equations of motion for two bodies in orbit around each other is a relatively straight-forward process. However, when a third mass is added, even if its mass is small, the equations of motion become impossible to solve symbolically. Among the complicating factors are the additional differential equations associated with the third body, as well as the fact that the system can no longer be defined in co-planar frame of reference. One strategy to approach the problem is to instead consider the restricted three body problem. In the restricted three body problem, the third object is considered to have no mass, and thus exerts no force on the other two bodies. As such, they will orbit as they would in the two body problem. Then, we can rotate our reference frame with the two larger bodies. While this simplifies the problem, it also introduces new parameters to consider, as this new reference frame is non-inertial. These are the centrifugal force and Coriolis Force. Taking these into account, we find the equation for the third body in the non-inertial reference frame to be:

$$\ddot{\vec{r_3}} = -\nabla\left[\frac{GM_1}{||\vec{r_3}-\vec{r_1}||} + \frac{GM_2}{||\vec{r_3}-\vec{r_2}||} + \frac{1}{2}\omega^2\vec{r_3}^{\,2}\right] + 2\vec{\omega}\times\vec{v_3} \tag{1}$$

The component of this equation that is operated on by the del operator is called the effective potential. We can plot one dimension of the potential and force in a two dimensional cross section, two dimensions on a three dimensional surface, or we can show them in three dimensional space by showing how the two dimensional surface change as we iterate through z-space.

2 Lagrange Points

Due to the effective potential and forces manifested in the rotating reference frame, there are five points in the system where there is no net fore acting on an object, called the Lagrange points. Recall that the forces are related to the gradient of the effective potential; thus the Lagrange points can also be described as the points of local minima or maxima in the effective potential. There are approximations that exist when the mass ratio $\frac{M_2}{M_1} = q$ is so small that the center of mass can be approximated to be at the center of M_1. However, when

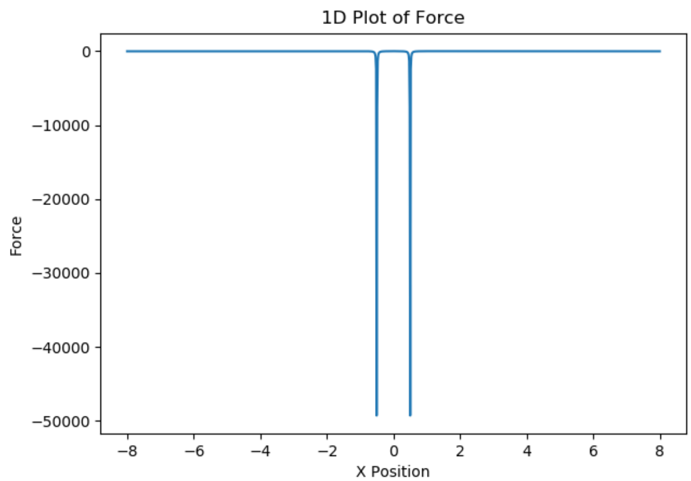

Figure 1: Plot of the force as a function of x.

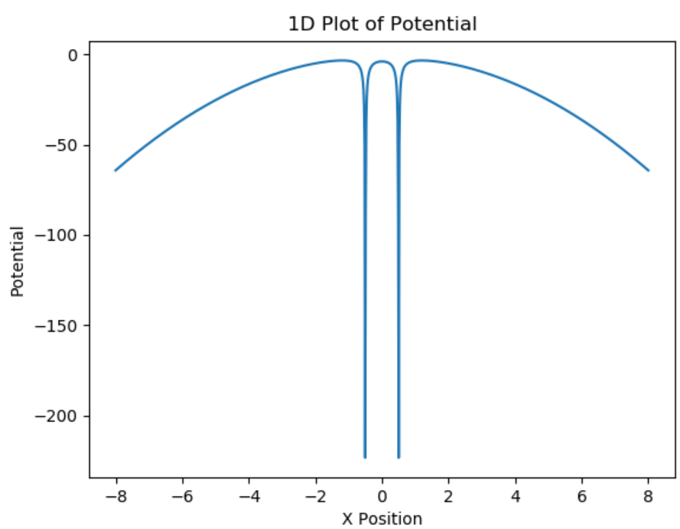

Figure 2: Plot of the effective potential as a function of x.

this is not the case, such as in a system a $q = 1$, The first three Lagrange points can be solved for by numerically determining the roots of a quintic polynomial. The fourth and fifth points can be more easily solved for using the geometry of the system. These points are easy to pick out on a contour map of the effective potential.

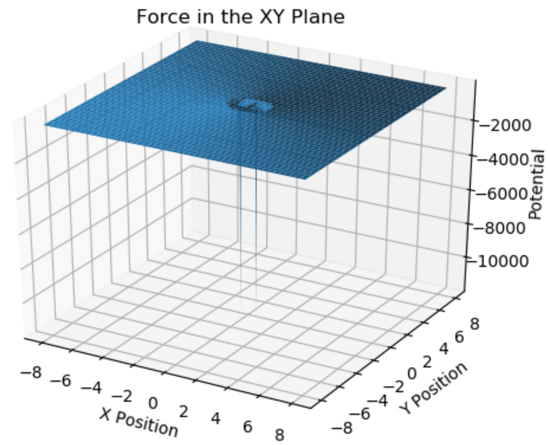

Figure 3: Plot of the force over the xy plane.

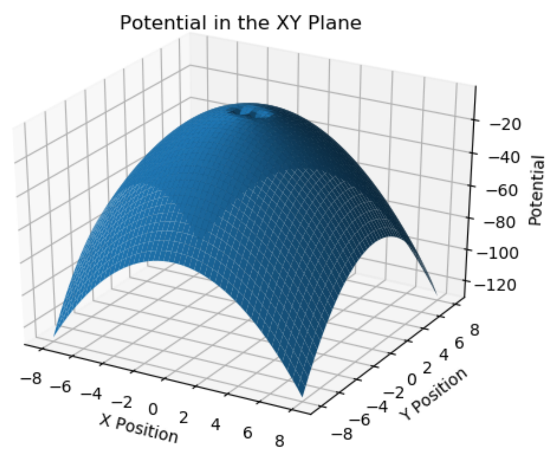

Figure 4: Plot of the potential over the xy plane.

Mathematically, the L4 and L5 points should be stable if $\frac{M_2}{M_1}$ is less than $\frac{1}{25}$. I was unable to get stable L4 and L5 points with mass ratios more than $\frac{1}{60}$.

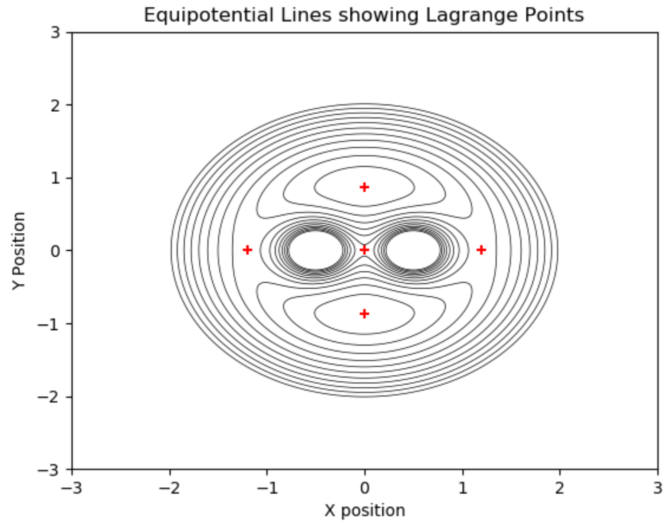

Figure 5: Plot of the Lagrange Points. L1 lies at the center, L2 lies on the positive x axis, L3 on the negative x axis, L4 in the positive y region and L5 in the negative y region.

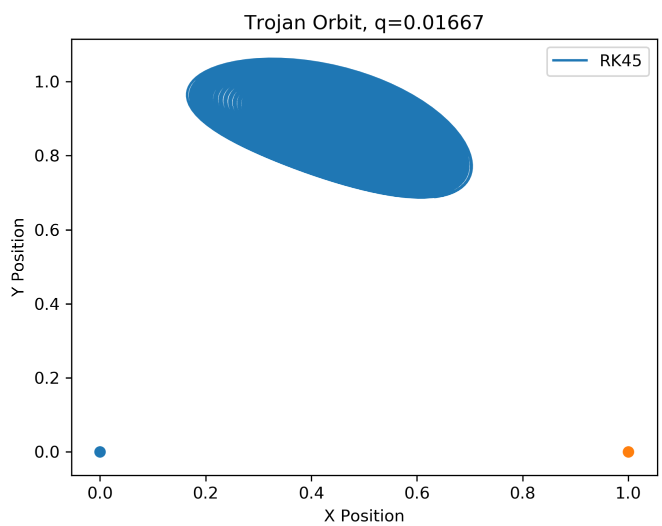

Figure 6: Stability of L4 when q=$\frac{1}{60}$

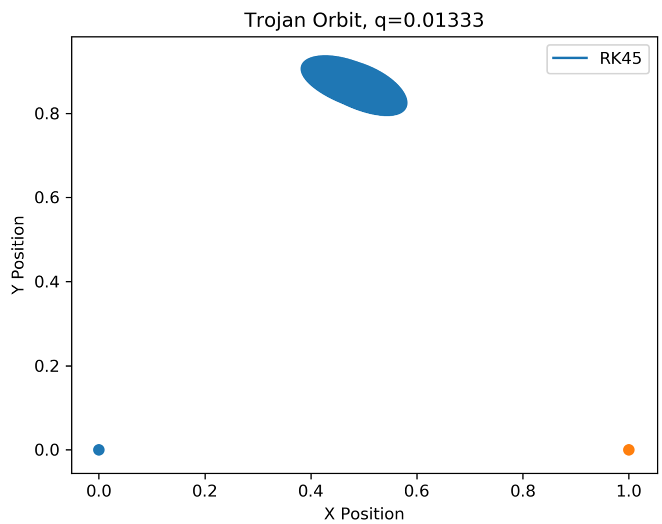

Figure 7: Stability of L4 when $q = \frac{1}{75}$

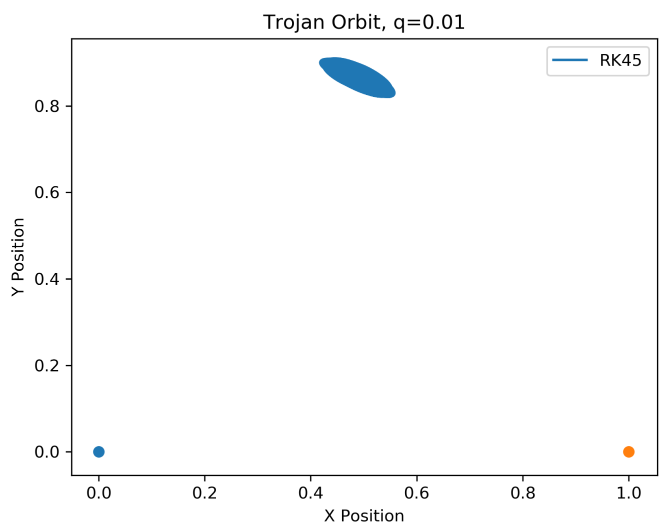

Figure 8: Stability of L4 when q=$\frac{1}{100}$

3 Trojan Asteroids

The Trojan asteroids exist at the L4 and L5 Lagrange points of the Sun-Jupiter system. They are called Trojans because they were first observed migrating between L4 and L5. In the rotating reference frame, these asteroid trace out a horsehoe pattern as they move from L4 to L5 and back. L4 and L5 are stable Lagrange points; we would not expect the asteroids to be sensitive to initial conditions, as the mass ratio of the Sun-Jupiter system exceeds that necessary to maintain stable L4 and L5 points. In fact, asteroid were created with a range of initial conditions that created these horseshoe shaped orbits.

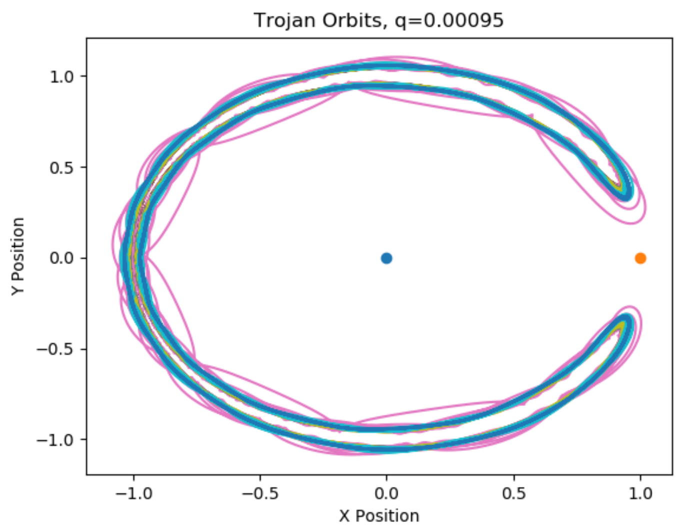

Figure 9: Horseshoe Orbits traced out by Trojan Asteroids.

I also attempted to plot the stable orbit of the JWST in the L2 point. L2 is an unstable point, and the JWST will have to correct its orbit over the course of its lifetime to remain there. My plot reflects this instability.

4 Roche Lobes

Binary star systems with mass ratios close to 1 and a small distance separating the bodies will deform each other, resulting in a bulging star. These bulges trace out equipotential surfaces. We can visualize the stars' shapes in three dimensions by iterating through z space and tracing a single equipotential line. In the one I created, there is a second outer equipotential ring that is unrelated to Roche Lobes. The stars' shapes can clearly be seen emerging from a single point below the xy plane, growing to a full cross section at the xy plane, and then shrinking back down.

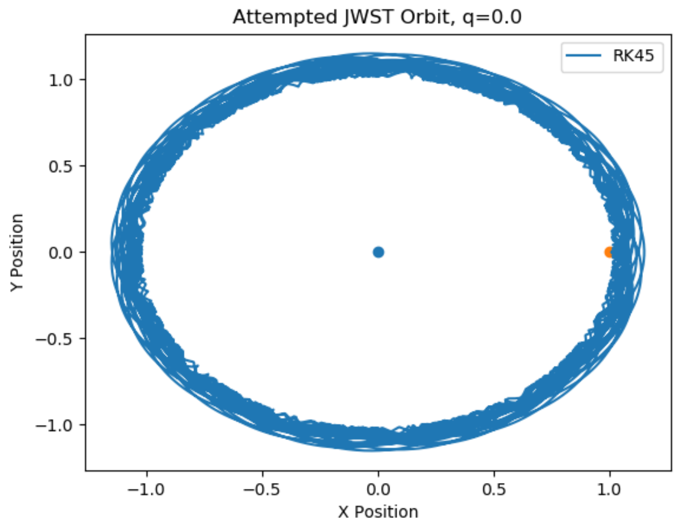

Figure 10: Attempted Plot of JWST

5 Non-Restricted 3 Body

The crux of this project thus far has been the assumption that the third mass in the three body system is negligible, and thus exerts no force on the two large masses. However, if the third mass approaches the magnitude of the two larger masses, this is no longer valid. To include the first body we can "simply" add to the differential equations required to solve the two body problem. We will need to ass z positions to all three masses, and the force equations due to and acting on the third mass. However, this system is in no way stable, and in actually exhibits chaotic behavior. This is clear in the plots of the orbits. Interestingly, in one instance, two of the bodies orbited and the third body went on its on trajectory, only for them to re-collide and have the third body switch with one of the first.

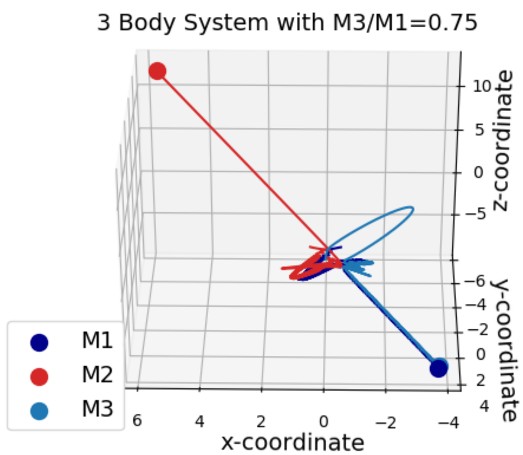

Figure 11: Chaotic orbits with a recapture event.

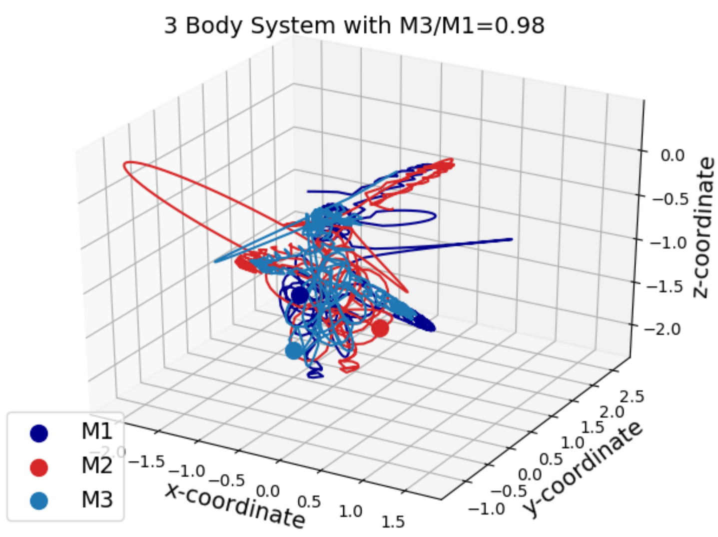

Figure 12: Chaotic orbits.