Modeling 3

Ryan Swope

April 20, 2021

1 The Feigenbaum Series

The Feigenbaum Series is an iterative function that was initially created to describe the population of a species given a carrying capacity of its environment. It is defined as:

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

Where x is bounded by $[0, 1]$ and r is bounded by $[0, 4]$. The Feigenbaum Series has three notable regions:

1. $0 \leq r \leq 1 : x_n$ approaches 0 as n approaches infinity.

2. $1 \leq r \leq 3 : x_n$ approaches $\frac{r-1}{r}$ as n approaches infinity.

3. $3 \leq r \leq 4 : x_n$ exhibits chaotic behavior. The Feigenbaum Series can be displayed in what is known as the Logistic Map. In the chaotic realm of the Logistic Map, the Feigenbaum Series undergoes bifurcation, where the series splits into two self-similar branches. The first four of these points are marked with the verticle lines in 1. The ratio of the r values at which the bifurcations occur creates the Feigenbaum Constant, which approximately approaches 4.669. Due to the discrete nature of the mapping seen here, the Feigenbaum Constant calculated from the Logistic Map was approximately 5.

   Due to the self similarity of the bifurcation mapping, the Logistic Map is considered a fractal. It is possible to demonstrate this by simply re-scaling two regions of bifuraction.

   However, there is perhaps a more elegant way to justify its fractal label. Fractals are shapes with non-integer dimensions. The Logistic Map is a plot of one-dimensional lines in 2D space. However, in the chaotic region the points included in the map become arbitrarily close to each other, and thus can be thought to cover an area. A one-dimensional object cannot span 2D space; thus, it is designated to have a dimension between 1 and 2!

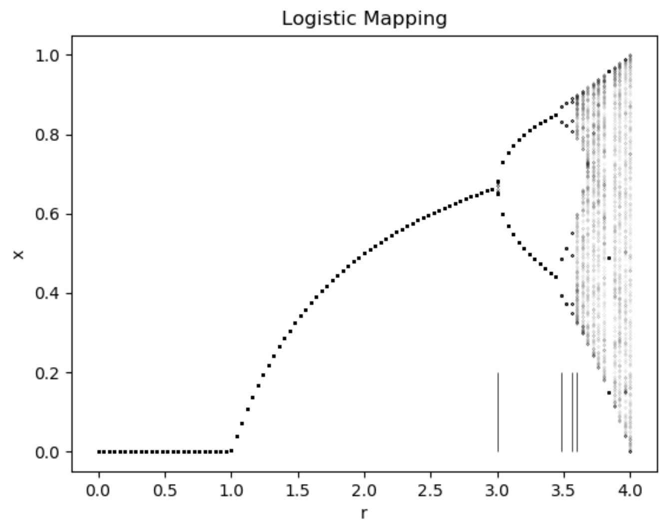

Figure 1: Logistic Map of the Feigenbaum Series

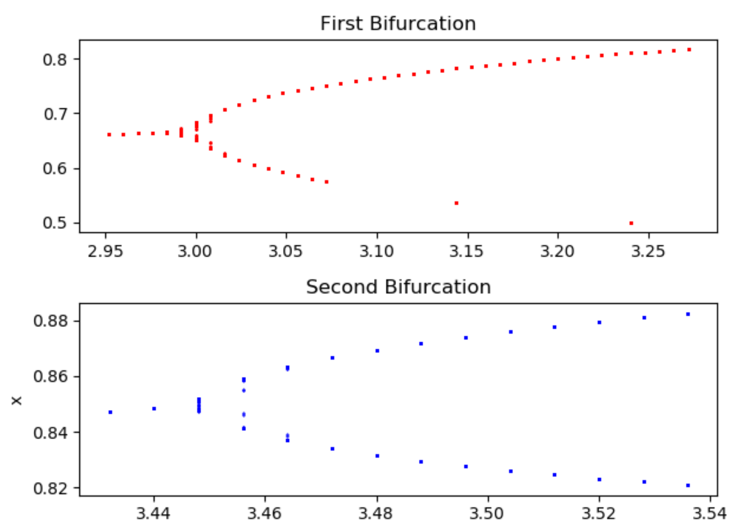

Figure 2: Fractal Nature of the Logistic Map. The difference in the number of points is simply due to the discrete nature of the iterative function.

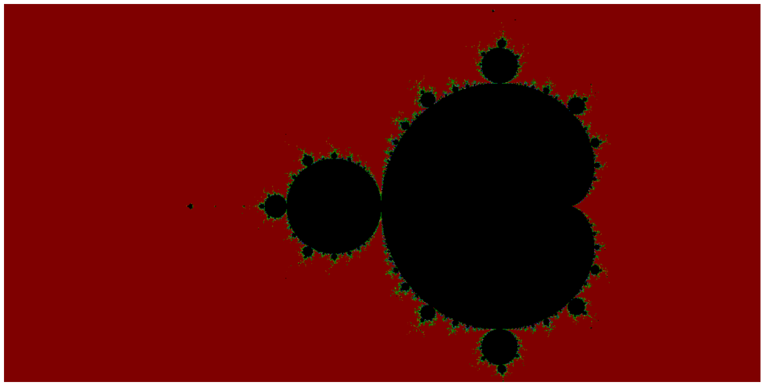

Figure 3: The Mandelbrot Set for $z = 0$, $c = x + iy$ and a bounding value of 2.

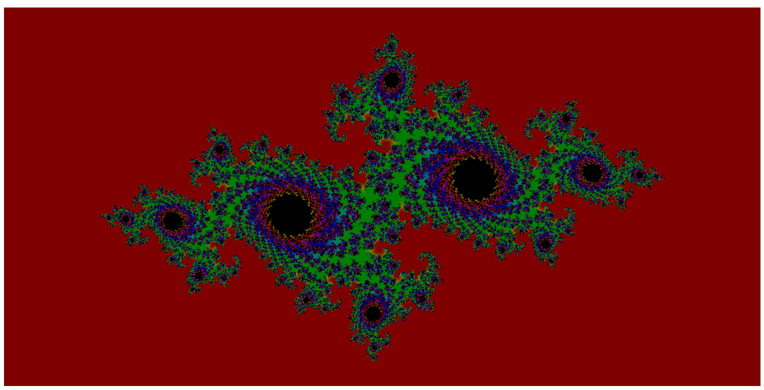

Figure 4: The Julia Set for $z = x + iy$, $c = -0.7269 + 0.1889i$ and a bounding value of 2.

2 The Mandelbrot Set

The Mandelbrot Set is the set of complex values satisfying bounded within the transformation:

$$f(z) = z^2 + c \mid z, c \in \mathbb{C} \tag{2}$$

The map of the Mandelbrot Set is typically colored by the number of iterations required for a the values so become unbounded for a given value. The black regions indicate areas infinitely bounded within the threshold.

The closely related Julia sets study the same equation, while holding z fixed and iterating through values of c. This can be accomplished by iterating through a finite region of complex space.

Figure 5: Original image

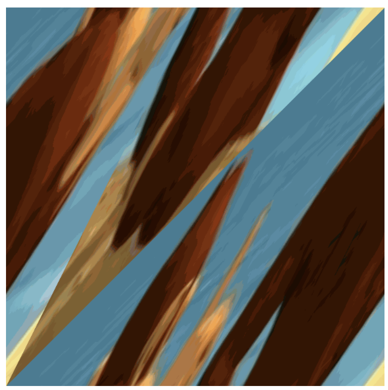

Figure 6: Image after one iteration of the Arnold's Cat Function.

3 Arnold's Cat (Ryan's Kong)

Arnold's Cat is a mapping of points inside the 2D region bounded by $x = [0, 1]$ and $y = [0, 1]$ defined as:

$$\Gamma : (x, y) \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \%1 \tag{3}$$

This transformation stretches points into the region bounded by $x = [0, 3]$ and $y = [0, 2]$ and then folds them back into the original region. Iterating this many times on a matrix or image will eventually recover the original matrix or image; in fact, this technique is used in cryptography to encrypt data.

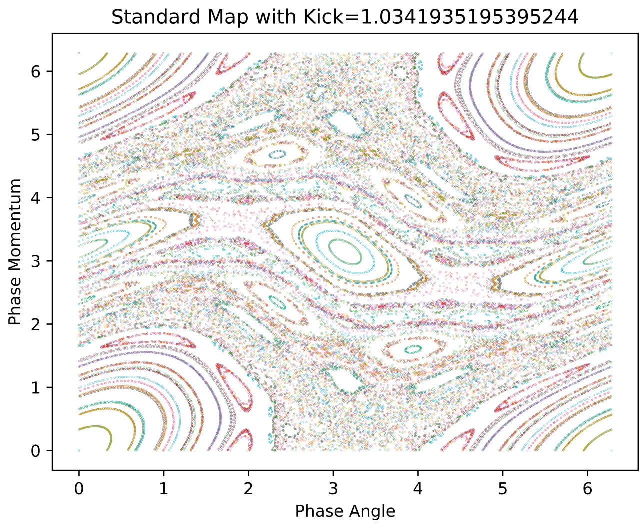

Figure 7: Kicked Rotator in Phase Space, with K=1.034.

4 Taylor-Greene-Chirikov Map

The Taylor-Greene-Chirikov map, more commonly known as the Standard map, is the Hamiltonian of a rotator that is kicked regularly as plotted in phase space. By plotting in phase space we are able to visually find islands where the rotator never visits in phase space; these islands shrink as K, the kick parameter approaches 2π.

5 Zaslavsky Map

The Zaslavski map is the Standard map with attenuation affecting the momentum, causing motion to slow and momentum to eventually reach 0. This is achieved by multiplying the momentum with an exponential decay factor. When plotted in phase space, the phase orbits tend toward zero.

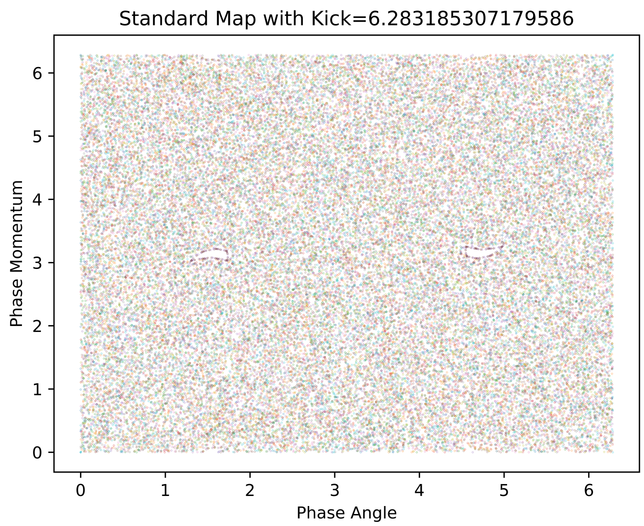

Figure 8: The fractal islands appear to disappear as the Kick Parameter approaches 2π. However, two islands are still clearly visible. It's possible this is an artifact of the resolution of the possible points.

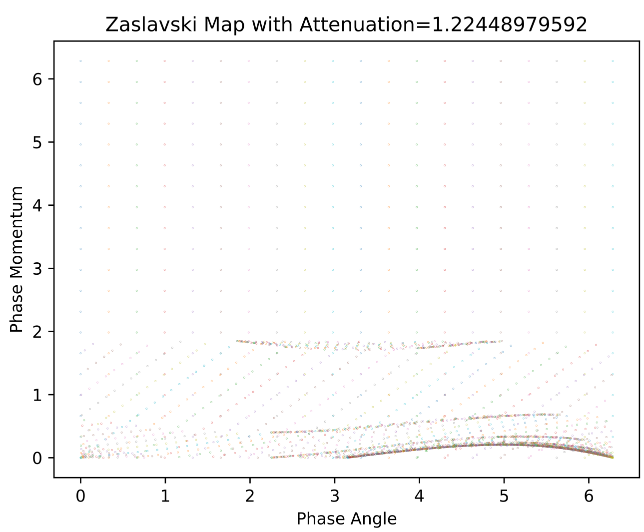

Figure 9: Kicked Rotator with attenuation.